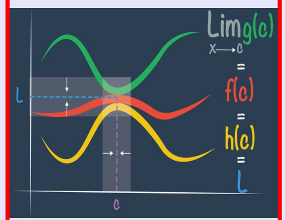
Math 261

Fall 2022

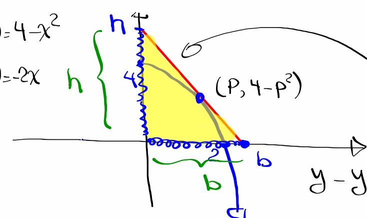
Lecture 34



What is the $\boxed{\text{Smallest possible area}}$ of the triangle is cut off by QI whose hypotenuse is tangent to the parabola $y=4-x^2$?

$f(x)=4-x^2$

$f'(x)=-2x$



$A=\dfrac{x\text{-int}\cdot y\text{-int}}{2}$

$m=f'(p)$

$m=-2p$

$y-y_1=m(x-x_1)$

$y-(4-p^2)=-2p(x-p)$

$y-4+p^2=-2px+2p^2$

$\boxed{y=-2px+p^2+4}$

x-Int.

$y=0$

$-2px+p^2+4=0$

$p^2+4=2px$

$x=\dfrac{p^2+4}{2p}$ ← x of x-Int.

Y-Int

$x=0$

$y=-2p(0)+p^2+4$ $\quad y=p^2+4$ ← Y of Y-Int.

$$\text{Area}=\frac{\frac{p^2+4}{2p}\cdot(p^2+4)}{2}=\frac{(p^2+4)(p^2+4)}{2p\cdot 2}=\frac{p^4+8p^2+16}{4p}$$

$$A(p)=\frac{p^4}{4p}+\frac{8p^2}{4p}+\frac{16}{4p}$$

$$A(p)=\frac{p^3}{4}+2p+\frac{4}{p}$$

optimize this to find smallest area.

$A(P) = \frac{P^3}{4} + 2P + \frac{4}{P}$ $A'(P) = \frac{3}{4}P^2 + 2 - \frac{4}{P^2}$

$A''(P) = \frac{6}{4}P + \frac{8}{P^3}$

$A'(P) = 0$ or $A'(P)$ undefined at $P=0$ — where Max & Min happens

$\frac{3}{4}P^2 + 2 - \frac{4}{P^2} = 0$

LCD $4P^2$

$3P^4 + 8P^2 - 16 = 0$

$$P^2 = \frac{-8 \pm \sqrt{64-4(3)(-16)}}{2(3)} = \frac{-8 \pm \sqrt{256}}{6} = \frac{-8 \pm 16}{6}$$

$P^2 = \frac{-8+16}{6}$ $\cancel{P^2 = \frac{-8-16}{6}} = -$

$P^2 = \frac{8}{6} = \frac{4}{3}$ $A''\left(\frac{2}{\sqrt{3}}\right) = \frac{6}{4}\left(\frac{2}{\sqrt{3}}\right) + \frac{8}{\left(\frac{2}{\sqrt{3}}\right)^3} = +$

$P = \frac{2}{\sqrt{3}}$ when $A''(P) > 0$

$$A\left(\frac{2}{\sqrt{3}}\right) = \frac{\left(\frac{2}{\sqrt{3}}\right)^3}{4} + 2\left(\frac{2}{\sqrt{3}}\right) + \frac{4}{2/\sqrt{3}}$$ Min. Value

$= \boxed{\frac{32\sqrt{3}}{9}}$ Smallest possible area

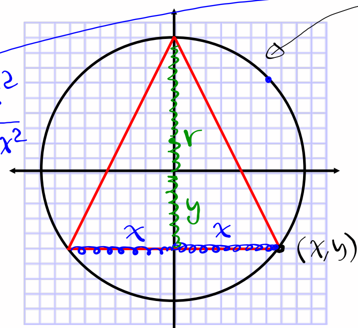
Please check

Find the dimensions of the isosceles triangle of largest area that can be inscribed in a circle with radius r.

$x^2 + y^2 = r^2$

$y^2 = r^2 - x^2$

$y = \sqrt{r^2 - x^2}$



base $= 2x$

height $= y + r$

Area $= \frac{bh}{2}$

$= \frac{2x \cdot (y+r)}{2}$

Area $= x(y+r)$

$A(x) = x\left(\sqrt{r^2-x^2} + r\right)$

$A'(x) = 1\left(\sqrt{r^2-x^2} + r\right) + x\left(\frac{-x}{\sqrt{r^2-x^2}}\right)$

$\frac{d}{dx}\left(\sqrt{r^2-x^2}\right) = \frac{d}{dx}\left[(r^2-x^2)^{1/2}\right]$

$\frac{1}{2}\cdot(r^2-x^2)^{-1/2}\cdot -2x$

$A'(x) = \sqrt{r^2-x^2} + r - \frac{x^2}{\sqrt{r^2-x^2}}$

$$A'(x) = \frac{r^2 - x^2 + r\sqrt{r^2-x^2} - x^2}{\sqrt{r^2-x^2}} = \frac{r^2 - 2x^2 + r\sqrt{r^2-x^2}}{\sqrt{r^2-x^2}}$$

$A'(x) = 0$

$r^2 - 2x^2 + r\sqrt{r^2-x^2} = 0$

$r\sqrt{r^2-x^2} = 2x^2 - r^2$ Square both Sides

$r^2(r^2-x^2) = 4x^4 - 4x^2r^2 + r^4$

$\cancel{r^4} - r^2x^2 = 4x^4 - 4x^2r^2 \cancel{+r^4}$

$4x^4 - 3x^2r^2 = 0$

$x^2(4x^2 - 3r^2) = 0$

$x = 0$ Not applicable

$x = \frac{r\sqrt{3}}{2}$

we can verify $f'(x)$ $+$ $\frac{r\sqrt{3}}{2}$ $-$

y+r

$h = y+r$

$\boxed{h = \frac{3}{2}r}$

$2x = r\sqrt{3}$

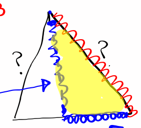
$x^2 + y^2 = r^2$

$\left(\frac{r\sqrt{3}}{2}\right)^2 + y^2 = r^2$

$\frac{3r^2}{4} + y^2 = r^2$

$y^2 = \frac{r^2}{4}$

$y = \frac{r}{2}$



$\frac{3r}{2}$

$\frac{r\sqrt{3}}{2}$

Find each Side?

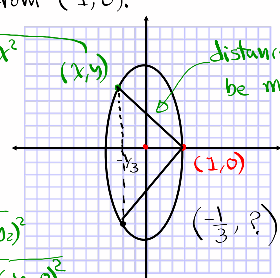
$S^2 = \left(\frac{r\sqrt{3}}{2}\right)^2 + \left(\frac{3r}{2}\right)^2$

$= \frac{r^2 \cdot 3}{4} + \frac{9r^2}{4} = \frac{12r^2}{4} = 3r^2 \to \boxed{S = r\sqrt{3}}$

Find (a point) [all points] on $4x^2 + y^2 = 4$ that is the farthest point from $(1, 0)$.

$4x^2 + y^2 = 4 \to y^2 = 4 - 4x^2$

$\frac{x^2}{1} + \frac{y^2}{4} = 1$



$\left(-\frac{1}{3}, ?\right), \left(\frac{1}{3}, ?\right)$

distance formula

$d = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

$= \sqrt{(x-1)^2 + (y-0)^2}$

$d = \sqrt{(x-1)^2 + y^2}$

Maximize $(x-1)^2 + y^2$

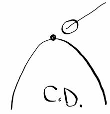
Max. $(x-1)^2 + 4 - 4x^2$

$f(x) = (x-1)^2 + 4 - 4x^2$

$f'(x) = 2(x-1)\cdot 1 - 8x$

$f'(x) = -6x - 2$

$f''(x) = -6 < 0$



$f'(x) = 0$

$-6x - 2 = 0$

$x = -\frac{1}{3}$

$4x^2 + y^2 = 4$

$4\left(-\frac{1}{3}\right)^2 + y^2 = 4$

$\frac{4}{9} + y^2 = 4$

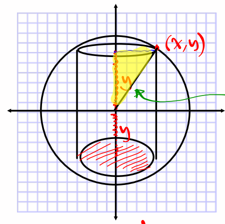
$y^2 = 4 - \frac{4}{9}$

$y^2 = \frac{32}{9}$ $y = \pm\frac{4\sqrt{2}}{3}$

$\left(-\frac{1}{3}, \frac{4\sqrt{2}}{3}\right), \left(-\frac{1}{3}, -\frac{4\sqrt{2}}{3}\right)$

A right circular cylinder is incribed in a sphere of radius r.

Find the largest possible volume of such cylinder.



height = $2y$

radius = x

Radius of sphere r

$x^2 + y^2 = r^2$

$x = \sqrt{r^2 - y^2}$

Volume of cylinder

= Area of base · height

$= \pi(\text{Radius})^2 \cdot 2y$

$= \pi\left(\sqrt{r^2 - y^2}\right)^2 \cdot 2y$

$V(y) = \pi(r^2 - y^2) \cdot 2y$

$= 2\pi\left[r^2 y - y^3\right]$

$V'(y) = 2\pi\left[r^2 - 3y^2\right]$

$V''(y) = 2\pi \cdot -6y$

for $y > 0$

$V''(y) < 0$

Max. Volume

$V'(y) = 0$

$r^2 - 3y^2 = 0 \rightarrow y = \frac{r}{\sqrt{3}}$

$V\left(\frac{r}{\sqrt{3}}\right) = 2\pi\left(r^2 \cdot \frac{r}{\sqrt{3}} - \left(\frac{r}{\sqrt{3}}\right)^3\right)$

$= 2\pi\left[\frac{3r^3}{3\sqrt{3}} - \frac{r^3}{3\sqrt{3}}\right] = 2\pi \cdot \frac{2r^3}{3\sqrt{3}}$

Largest Volume $= \frac{4\pi r^3}{3\sqrt{3}}$ unit^3.

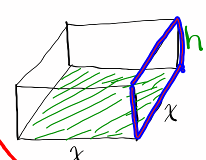
You have 1200 cm^2 of flat material, and you wish a make open top box with square base. $V = 20 \cdot 20 \cdot 10$ (Volume 4000 cm^3)

Find the largest possible volume. $x > 0$

$V = LWH$

$V = x \cdot x \cdot h$

$V = x^2 h$



Bottom + 4 Sides

$x^2 + 4 \cdot xh = 1200$

$4xh = 1200 - x^2$

$h = \frac{1200 - x^2}{4x}$

$h = \frac{1200 - 20^2}{4(20)} = 10$

$V(x) = \cancel{x}^2 \cdot \frac{1200 - x^2}{4\cancel{x}}$

$V(x) = \frac{x(1200 - x^2)}{4}$

$V(x) = \frac{1}{4}\left(1200x - x^3\right)$

$V'(x) = \frac{1}{4}\left(1200 - 3x^2\right)$

$V''(x) = \frac{1}{4}(-6x)$

for $x > 0$ $V''(x) < 0$ Max

$V'(x) = 0$

$1200 - 3x^2 = 0$

$\boxed{x = 20}$